

Plancherel Theorem and the Left Ideals of the Group Algebra for the Jacobi Group.

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Abstract

Let $G = SL(2, \mathbb{R})$ be the 2×2 connected real semisimple Lie group and let KAN be the Iwasawa decomposition of $SL(2, \mathbb{R})$. Let $J = H \rtimes SL(2, \mathbb{R})$ be the Jacobi group, which is the semidirect product of the two groups H with $SL(2, \mathbb{R})$. It plays an important role in Quantum Mechanics. The purpose of this paper is to define the Fourier transform in order to obtain the Plancherel theorem for the group J . To this end a classification of all left ideals of the group algebra $L^1(H \rtimes AN)$.

Keywords: Jacobi Group, Iwasawa Decomposition, Fourier Transform and Plancherel Theorem, Left Ideals

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1 Introduction

1.1. The Jacobi group the semidirect product of the Heisenberg and the symplectic group $SL(2, \mathbb{R})$ plays an important role in quantum mechanics. In Quantum optics represent a physical realization of the coherent states associated to the Jacobi group. The Jacobi group is responsible for the squeezed states and has an important object in quantum mechanics, geometric quantization, optics. Abstract harmonic analysis is the field of the most modern branches of harmonic analysis, having its roots in the mid-twentieth century, is analysis on topological groups. If the group is neither abelian nor compact, no general satisfactory theory is currently known.

1.2. First In this paper I will define the Fourier transform in order to establish the Plancherel formula on the Jacobi group $H \rtimes SL(2, \mathbb{R})$, where H is the 3-dimensional Heisenberg group and

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} X & a & b \\ & c & d \end{pmatrix} : \det X = 1 \right\} \quad (1)$$

Secondly, I will give classification for all left ideals of the group algebra $L^1(H \rtimes AN)$, where AN is the solvable Lie group in the Iwasawa decomposition KAN of $SL(2, \mathbb{R})$.

2 Fourier Transform and Plancherel Formula on $SL(2, \mathbb{R})$

2.1. In the following and far away from the representations theory of Lie groups we use the Iwasawa decomposition of $SL(2, \mathbb{R})$, to define the Fourier transform and to demonstrate Plancherel formula on the connected real semisimple Lie group $SL(2, \mathbb{R})$. Therefore let $SL(2, \mathbb{R})$ be the real Lie group, which is

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d) \in \mathbb{R}^4 \text{ and } ad - bc = 1 \right\} \quad (2)$$

and let $SL(2, \mathbb{R}) = KNA$ be the Iwasawa decomposition of $SL(2, \mathbb{R})$, where

$$\begin{aligned} K &= \left\{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} = SO(2) : \phi \in \mathbb{R} \right\} \\ N &= \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{R} \right\} \\ A &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_+^* \right\} \end{aligned} \quad (3)$$

Hence every $g \in SL(2, \mathbb{R})$ can be written as $g = kan \in SL(2, \mathbb{R})$, where $k \in K$, $a \in A$, $n \in N$.

2.2. We denote by $L^1(G)$ the Banach algebra that consists of all complex valued functions on the group G , which are integrable with respect to the Haar measure dg of G and multiplication is defined by convolution product on G , where $G = SL(2, \mathbb{R})$. And denote by $L^2(G)$ the Hilbert space of G . So we have for any $f \in L^1(G)$ and $\phi \in L^1(G)$

$$\phi * f(h) = \int_G f(g^{-1}h) \phi(g) dg \quad (4)$$

The Haar measure dg on a connected real semi-simple Lie group $G = SL(n, \mathbb{R})$, can be calculated from the Haar measures dn , da and dk on N ; A and K ; respectively,

by the formula

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank)dadndk \quad (5)$$

Keeping in mind that $a^{-2\rho}$ is the modulus of the automorphism $n \rightarrow ana^{-1}$ of N we get also the following representation of dg

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank)dadndk = \int_N \int_A \int_K f(nak)a^{-2\rho}dndadk \quad (6)$$

where

$$\rho = 2^{-1} \sum_{\alpha \geq 0, \alpha \neq 0} m(\alpha)\alpha$$

and $m(\alpha)$ denotes the multiplicity of the root α see [17] or again ρ = the dimension of the nilpotent group N . Furthermore, using the relation $\int_G f(g)dg = \int_G f(g^{-1})dg$, we receive

$$\int_G f(g)dg = \int_K \int_A \int_N f(kan)a^{2\rho}dndadk \quad (7)$$

2.3. Let Γ be a connected compact Lie group and let \underline{k} be the Lie algebra of Γ . Let (X_1, X_2, \dots, X_m) a basis of \underline{k} , such that the both operators

$$\Delta = \sum_{i=1}^m X_i^2 \quad (8)$$

$$D_q = \sum_{0 \leq l \leq q} \left(- \sum_{i=1}^m X_i^2 \right)^l \quad (9)$$

are left and right invariant (bi-invariant) on Γ , this basis exist see [2, p.564]. For $l \in \mathbb{N}$, let $D^l = (1 - \Delta)^l$, then the family of semi-norms $\{\sigma_l, l \in \mathbb{N}\}$ such that

$$\sigma_l(f) = \int_\Gamma |D^l f(y)|^2 dy)^{\frac{1}{2}}, \quad f \in C^\infty(\Gamma) \quad (10)$$

define on $C^\infty(\Gamma)$ the same topology of the Frechet topology defined by the semi-norms $\|X^\alpha f\|_2$ defined as

$$\|X^\alpha f\|_2 = \int_\Gamma (|X^\alpha f(y)|^2 dy)^{\frac{1}{2}}, \quad f \in C^\infty(\Gamma) \quad (11)$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, see [2, p.565]

Let $\widehat{\Gamma}$ be the set of all equivalence classes of irreducible unitary representations of Γ . If $\gamma \in \widehat{\Gamma}$, we denote by E_γ the space of representation γ and d_γ its dimension then we get

Definition 2.1. The Fourier transform of a function $f \in C^\infty(\Gamma)$ is defined as

$$Tf(\gamma) = \int_{\Gamma} f(x) \gamma(x^{-1}) dx \quad (12)$$

where T is the Fourier transform on Γ

Theorem (A. Cerezo) 2.1. Let $f \in C^\infty(\Gamma)$, then we have the inversion of the Fourier transform

$$f(x) = \sum_{\gamma \in \widehat{\Gamma}} d\gamma \operatorname{tr}[Tf(\gamma) \gamma(x)] \quad (13)$$

$$f(I_\Gamma) = \sum_{\gamma \in \widehat{\Gamma}} d\gamma \operatorname{tr}[Tf(\gamma)] \quad (14)$$

and the Plancherel formula

$$\|f(x)\|_2^2 = \int_{\Gamma} |f(x)|^2 dx = \sum_{\gamma \in \widehat{\Gamma}} d_\gamma \|Tf(\gamma)\|_{H.S}^2 \quad (15)$$

for any $f \in L^1(\Gamma)$, where I_Γ is the identity element of Γ and $\|Tf(\gamma)\|_{H.S}^2$ is the Hilbert-Schmidt norm of the operator $Tf(\gamma)$

Fourier did not actually assume any underlying group structure or representation theory but we typically associate his work with the case of the circle group in the following form using complex exponentials

$$f(x) = \sum_{n=-\infty}^{\infty} Tf(n) e^{ixn} = \sum_{n=-\infty}^{\infty} c_n e^{ixn}, \quad n \in \mathbb{Z} \quad (16)$$

where

$$c_m = Tf(m) = \int_{SO(2)} f(x) e^{-ixm} dx \quad (17)$$

The group is $SO(2) = S^1$ or \mathbb{R}/\mathbb{Z} and the multiplicative characters are e^{ixn} , group homomorphisms from the circle $K = SO(2)$ to the multiplicative group of non-zero complex numbers. Fourier actually preferred to express the coefficients using what is now known as the Plancherel formula

$$\|f(x)\|_2^2 = \int_{SO(2)} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{\infty} |Tf(n)|^2 \quad (18)$$

where

$$S^1 = SO(2) = \left\{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} : \phi \in \mathbb{R} \right\} \quad (19)$$

Definition 2.2. For any function $f \in \mathcal{D}(G)$, we can define a function $\Upsilon(f)$ on $G \times K = G \times SO(2)$ by

$$\Upsilon(f)(g, k_1) = \Upsilon(f)(kna, k_1) = f(gk_1) = f(knak_1) \quad (20)$$

for $g = kna \in G$, and $k_1 \in K$. The restriction of $\Upsilon(f) * \psi(g, k_1)$ on $K(G)$ is $\Upsilon(f) * \psi(g, k_1) \downarrow_{K(G)} = f(nak_1) = f(g) \in \mathcal{D}(G)$, and $\Upsilon(f)(g, k_1) \downarrow_K = f(kna) \in \mathcal{D}(G)$

Remark 2.1. $\Upsilon(f)$ is invariant in the following sense

$$\Upsilon(f)(gh, h^{-1}k_1) = \Upsilon(f)(g, k_1) \quad (21)$$

Definition 2.3. If f and ψ are two functions belong to $\mathcal{D}(G)$, then we can define the convolution of $\Upsilon(f)$ and ψ on $G \times K = G \times S^1 = G \times SO(2)$ as

$$\begin{aligned} \Upsilon(f) * \psi(g, k_1) &= \int_G \Upsilon(f)(gg_2^{-1}, k_1) \psi(g_2) dg_2 \\ &= \int_{SO(2)} \int_N \int_A \Upsilon(f)(knaa_2^{-1}n_2^{-1}k^{-1}k_1) \psi(k_2n_2a_2) dk_2 dn_2 da_2 \end{aligned}$$

So we get

$$\begin{aligned} \Upsilon(f) * \psi(g, k_1) &\downarrow_{K(G)} = \Upsilon(f) * \psi(I_K na, k_1) \\ &= \int_{SO(2)} \int_N \int_A f(naa_2^{-1}n_2^{-1}k^{-1}k_1) \psi(k_2n_2a_2) dk_2 dn_2 da_2 \\ &= \Upsilon(f) * \psi(na, k_1) \end{aligned}$$

where $g_2 = k_2n_2a_2$

Definition 2.4. If $f \in \mathcal{D}(G)$ and let $\Upsilon(f)$ be the associated function to f , we define the Fourier transform of $\Upsilon(f)(g, k_1)$ by

$$\begin{aligned} \mathcal{F}\Upsilon(f)(I_{S^1}, \xi, \lambda, \gamma, I_{S^1}) &= \mathcal{F}\Upsilon(f)(I_{S^1}, \xi, \lambda, I_{S^1}) \\ &= \int_{S^1} \int_N \int_A \left[\sum_{l=-\infty}^{\infty} \int_{S^1} T\Upsilon(f)(kna, k_1) e^{-ilk} dk \right] a^{-i\lambda} e^{-i\langle \xi, n \rangle} e^{-imk_1} da dn dk_1 \\ &= \int_{S^1} \int_N \int_A [\Upsilon(f)(I_{S^1} na, k_1)] a^{-i\lambda} e^{-i\langle \xi, n \rangle} e^{-imk_1} da dn dk_1 \quad (22) \end{aligned}$$

where \mathcal{F} is the Fourier transform on AN and T is the Fourier transform on $SO(2)$, and I_{S^1} is the identity element of $S^1 = SO(2)$

Plancherel's Theorem on the Group G 2.2. For any function $f \in L^1(G) \cap L^2(G)$, we get

$$\int_G |f(g)|^2 dg = \int_A \int_N \int_{S^1} |f(kna)|^2 da dn dk = \sum_{m=-\infty}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \|T\mathcal{F}f(\lambda, \xi, m)\|_2^2 d\lambda d\xi \quad (23)$$

$$f(I_A I_N I_{S^1}) = \int_N \int_A \sum_{m=-\infty}^{\infty} T \mathcal{F} f((\lambda, \xi, m)) d\lambda d\xi = \sum_{m=-\infty}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} T \mathcal{F} f(\lambda, \xi, m) d\lambda d\xi \quad (24)$$

where I_A, I_N , and I_K are the identity elements of A , N and K respectively, where \mathcal{F} is the Fourier transform on AN and T is the Fourier transform on K , and I_K is the identity element of K

Proof: First let $\overset{\vee}{f}$ be the function defined by

$$\overset{\vee}{f}(kna) = \overline{f((kna)^{-1})} = \overline{f(a^{-1}n^{-1}k^{-1})} \quad (25)$$

Then we have

$$\begin{aligned} & \int_G |f(g)|^2 dg \\ &= \Upsilon(f) * \overset{\vee}{f}(I_{S^1} I_N I_A, I_{S^1}) \\ &= \int_G \Upsilon(f)(I_{S^1} I_N I_A(g_2^{-1}), I_{S^1}) \overset{\vee}{f}(g_2) dg_2 \\ &= \int_A \int_N \int_{S^1} \Upsilon(f)(a_2^{-1} n_2^{-1} k_2^{-1}, I_{S^1}) \overset{\vee}{f}(k_2 n_2 a_2) da_2 dn_2 dk_2 \\ &= \int_A \int_N \int_{S^1} f(a_2^{-1} n_2^{-1} k_2^{-1}) \overline{f((k_2 n_2 a_2)^{-1})} da_2 dn_2 dk_2 \\ &= \int_A \int_N \int_{S^1} |f(a_2 n_2 k_2)|^2 da_2 dn_2 dk_2 \end{aligned} \quad (26)$$

Secondly

$$\begin{aligned}
& \Upsilon(f) * \check{f}(I_{S^1} I_N I_A, I_{S^1}) \\
= & \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{F}(\Upsilon(f) * \check{f})(I_{S^1}, \lambda, \xi, I_{S^1}) d\lambda d\xi \\
= & \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \int_{S^1} \Upsilon(f) * \check{f}(kna, k_1) e^{-ilk} dka^{-i\lambda} e^{-i\langle \xi, n \rangle} e^{-imk_1} dadndk_1 d\lambda d\xi \\
= & \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \Upsilon(f) * \check{f}(I_{S^1} na, k_1) e^{-ilk} dka^{-i\lambda} e^{-i\langle \xi, n \rangle} e^{-imk_1} dadndk_1 d\lambda d\xi \\
= & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_{S^1} \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} \Upsilon(f)(I_{S^1} naa_2^{-1} n_2^{-1} k_2^{-1}, k_1) \check{f}(k_2 n_2 a_2) e^{-imk_1} dk_1 \\
& dndadk_2 dn_2 da_2 a^{-i\lambda} e^{-i\langle \xi, n \rangle} d\lambda d\xi \\
= & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_{S^1} \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} f(naa_2^{-1} n_2^{-1} k_2^{-1} k_1) \check{f}(k_2 n_2 a_2) e^{-imk_1} dk_1 dk_2 \\
& a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndadn_2 da_2 d\lambda d\xi
\end{aligned}$$

where

$$e^{-i\langle \xi, n \rangle} = e^{-i\xi n} \quad (27)$$

Using the fact that

$$\int_A \int_N \int_{S^1} f(kna) dadndk = \int_N \int_A \int_{S^1} f(kan) a^2 dndadk \quad (28)$$

and

$$\begin{aligned}
& \int_{\mathbb{R}} \int_A \int_N \int_{S^1} f(kna) e^{-i\langle \xi, n \rangle} dadndk d\xi \\
= & \int_{\mathbb{R}} \int_A \int_N \int_{S^1} f(kan) e^{-i\langle \xi, an_1 a^{-1} \rangle} a^2 dadndk d\xi \\
= & \int_{\mathbb{R}} \int_A \int_N \int_{S^1} f(kan) e^{-i\langle a\xi a^{-1}, n \rangle} a^2 dadndk d\xi \\
= & \int_{\mathbb{R}} \int_A \int_N \int_{S^1} f(kan) e^{-i\langle \xi, n \rangle} dadndk d\xi \quad (29)
\end{aligned}$$

Then we get

$$\begin{aligned}
& \Upsilon(f) * \check{f}(I_{S^1} I_N I_A, I_{S^1}) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_{S^1} \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} f(na_2^{-1} n_2^{-1} k_2^{-1}, k_1) \check{f}(k_2 n_2 a_2) e^{-imk_1} dk_1 dk_2 \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndadn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{S^1} f(aa_2^{-1} nn_2^{-1} k_2^{-1}, k_1) \check{f}(k_2 n_2 a_2) e^{-imk_1} dk_1 dk_2 \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndadn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{S^1} f(ank_2^{-1}, k_1) \check{f}(k_2 n_2 a_2) e^{-imk_1} dk_1 dk_2 \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndadn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{S^1} f(ank_2^{-1} k_1) \check{f}(k_2 n_2 a_2) e^{-imk_1} dk_1 dk_2 \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndadn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{S^1} f(ank_1^{-1}) \check{f}(k_2 n_2 a_2) e^{-imk_1} e^{-imk_2} dk_1 dk_2 \\
&\quad a^{-i\lambda} a_2^{-i\lambda} e^{-i\langle \xi, n+n_2 \rangle} dndadn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{S^1} f(ank_1^{-1}) \overline{f(a_2^{-1} n_2^{-1} k_2^{-1})} e^{-imk_1} e^{-imk_2} dk_1 dk_2 \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} a_2^{-i\lambda} e^{-i\langle \xi, n_2 \rangle} dndadn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{S^1} f(ank_1^{-1}) \overline{f(a_2 n_2 k_2)} e^{-imk_2} e^{-imk_1} dk_1 dk_2 \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} a_2^{-i\lambda} e^{i\langle \xi, n_2 \rangle} dndadn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_A \int_N \int_A \int_N \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{S^1} f(ank_1^{-1}) \overline{f(a_2 n_2 k_2)} e^{-imk_2} e^{-imk_1} dk_1 dk_2 \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} \overline{a_2^{-i\lambda} e^{-i\langle \xi, n_2 \rangle}} dndadn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} T\mathcal{F}f(\lambda, \xi, m) \overline{T\mathcal{F}f(\lambda, \xi, m)} d\lambda d\xi = \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} |T\mathcal{F}(f)(\lambda, \xi, m)|^2 d\lambda d\xi
\end{aligned}$$

3 Fourier Transform and Plancherel Formula H .

3.1. Let H be the real Heisenberg group of dimension $2n + 1$ which consists of all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix} \quad (30)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}$ and I is the identity matrix of order n .

Let $H = \mathbb{R}^{n+1} \rtimes_{\iota} \mathbb{R}^n$ be the group of the semi-direct product of the group \mathbb{R}^{n+1} and \mathbb{R}^n , via the group homomorphism $\iota : \mathbb{R}^n \rightarrow \text{Aut}(\mathbb{R}^{n+1})$, which is defined by:

$$\iota(x)(z, y) = (z + xy, y) = x(z, y) \quad (31)$$

for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $xy = \sum_{i=1}^n x_i y_i$, where $\text{Aut}(\mathbb{R}^{n+1})$ is the group of all automorphism of \mathbb{R}^{n+1}

3.2. Let $C^\infty(H)$, $\mathcal{D}(H)$, $\mathcal{D}'(H)$, $\mathcal{E}'(G)$ respectively the space of C^∞ - functions, C^∞ with compact support, distribution and distribution with compact support on G . The Schwartz space $\mathcal{S}(G)$ of G can be considered as the Schwartz space $\mathcal{S}(\mathbb{R}^{2n+1})$ of the vector group \mathbb{R}^{2n+1} . The action ι of the group \mathbb{R}^n on \mathbb{R}^{n+1} defines a natural action ι on the dual $(\mathbb{R}^n)^*$ of the group $\mathbb{R}^{n+1}((\mathbb{R}^{n+1})^* \simeq \mathbb{R}^{n+1})$ which is given by :

$$x(\eta, \lambda) = (\eta, \eta x + \lambda)$$

for any $\lambda \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$, where ;

$$x(\eta, \lambda) = \iota(x)(\eta, \lambda)$$

and

$$\eta x = \sum_{i=1}^n \eta x_i$$

Definition 3.1. For every $f \in \mathcal{S}(G)$, one can define its Fourier transform $\mathcal{F}f$ by :

$$\mathcal{F}f(\xi) = \int_G f(X) e^{-i\langle \xi, X \rangle} dX \quad (32)$$

where $X = ((z, y); x) \in G$, $\xi = ((\eta, \lambda); \mu) \in G$, and $dX = dz dy dx$ the Lebesgue measure on G

$$\langle \xi, X \rangle = z\eta + y\lambda + x\mu = z\eta + \sum_{i=1}^n \lambda_i y_i + \sum_{i=1}^n x_i \mu_i$$

It is clear that the function $\mathcal{F}f \in \mathcal{S}(G)$ and the mapping $f \mapsto \mathcal{F}f$ is a topological isomorphism vector space $\mathcal{S}(G)$ onto it self.

Theorem 3.1. *The Fourier transform \mathcal{F} satisfies :*

$$\overset{\vee}{g} * f(0) = \int_G \mathcal{F}f(\xi) \overline{\mathcal{F}g(\xi)} d\xi \quad (33)$$

for every $f \in \mathcal{S}(G)$ and $g \in \mathcal{S}(G)$, where $\overset{\vee}{g}(X) = \overline{g(X^{-1})}$, $\xi = ((\eta, \lambda); \mu)$, $d\xi = d\eta d\lambda d\mu$, is the Lebesgue measure on $G = \mathbb{R}^{2n+1}$, and $*$ denotes the convolution product on G

Proof : By the classical Fourier transform, we have:

$$\begin{aligned} \overset{\vee}{g} * f(0) &= \int_G \mathcal{F}(\overset{\vee}{g} * f)(\xi) d\xi \\ &= \int_G \int_G \overset{\vee}{g} * f(X) e^{-i\langle \xi, X \rangle} dX d\xi \\ &= \int_G \int_G \int_G f(Y^{-1}X) \overline{g(Y^{-1})} e^{-i\langle \xi, X \rangle} dY dX d\xi \\ &= \int_G \int_G \int_G f(YX) \overline{g(Y)} e^{-i\langle \xi, X \rangle} dY dX d\xi \end{aligned} \quad (34)$$

By change of variable $YX = X'$, with $X' = ((z, y); x)$ and $Y = ((z', y'); x')$ we get :

$$\begin{aligned} X &= Y^{-1}X' = ((-x'(-z', -y')) - x')((z, y); x) \\ &= ((-x'(z - z', y - y')); x - x') \end{aligned}$$

this gives us :

$$\begin{aligned} &e^{-i\langle \xi, X \rangle} \\ &= e^{-i\langle \xi, Y^{-1}X' \rangle} \\ &= e^{-i\langle (-x'(\eta, \lambda); \mu); ((z - z', y - y'); x - x') \rangle} \\ &= e^{-i\langle ((\eta, -\eta x' + \lambda); \mu), ((z - z', y - y'); x - x') \rangle} \end{aligned} \quad (35)$$

By the invariant of the Lebesgue measures $d\eta$, $d\lambda$, and $d\mu$ we obtain,

$$\begin{aligned}
\check{g} * f(0) &= \int_G f(X) \overline{g(X)} dX \\
&= \int_G \int_G \int_G f(X) e^{-i\langle \xi, X \rangle} \overline{g(Y) e^{-i\langle \xi, Y \rangle}} dX dY d\xi \\
&= \int_G \mathcal{F}f(\xi) \overline{\mathcal{F}g(\xi)} d\xi
\end{aligned} \tag{36}$$

where $0 = ((0, 0); 0)$ is the identity of G , whence the theorem.

Corollary 3.1. *In theorem 3.1, if we take $g = \check{f}$, we obtain the Plancherel formula on G*

$$\check{f} * f(0) = \int_G |f(X)|^2 dX = \int_{\mathbb{R}^n} |\mathcal{F}f(\xi)|^2 d\xi \tag{37}$$

4 Fourier Transform and Plancherel Formula on the Real Jacobi group $N \rtimes SL(2, \mathbb{R})$

4.1. Let H be the 3-dimensional Heisenberg group, with multiplication

$$(z_1, y_1, x_1)(z_2, y_2, x_2) = (z_1 + z_2 + x_1 y_2 - x_2 y_1, y_1 + x_1, x_2 + y_2) \tag{38}$$

The group H is isomorphic onto the following Heisenberg group of all matrices

$$N = \{ \mathbb{R}^2 \rtimes_{\sigma} \mathbb{R} \simeq \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array} \right) : (z, y, x) \in \mathbb{R}^3 \} \tag{39}$$

where $\sigma : \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^2)$ is the group homomorphism from the real group into the group $\text{Aut}(\mathbb{R}^2)$ of all automorphisms of the vector group \mathbb{R}^2 , defined as

$$\sigma(x)(z, y) = (z + xy, y)$$

So the group H can be identified with the group N , where the multiplication becomes as

$$(z_1, y_1, x_1)(z_2, y_2, x_2) = (z_1 + z_2 + x_1 y_2, y_1 + x_1, x_2 + y_2) \tag{40}$$

Now we define the Jacobi group J as $N \rtimes_{\rho} SL(2, \mathbb{R})$ the semidirect of the Heisenberg group N and the real semisimple Lie group $SL(2, \mathbb{R})$, where $\rho :$

$SL(2, \mathbb{R}) \rightarrow Aut(N)$ is the group homomorphism from the real group into the group $Aut(N)$ of all automorphisms of the vector group N , defined as

$$\begin{aligned}\rho(M)(z, y, x) &= (z, \begin{bmatrix} y & x \end{bmatrix} M) \\ &= (z, \begin{bmatrix} y & x \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}) \\ &= (z, ya + xc, yb + xd)\end{aligned}$$

where $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So, any element $g \in J$ can be written in a unique way as $g = (X, M)$ with $M \in SL(2, \mathbb{R})$ and $X = (z, y, x) \in N$. Multiplication in J is then given as

$$\begin{aligned}(X_1, M_1)(X_2, M_2) &= (z_1, y_1, x_1, M_1)(z_2, y_2, x_2, M_2) \\ &= ((z_1, y_1, x_1)(z_2, \begin{bmatrix} y_2 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}), \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}) \\ &= ((z_1, y_1, x_1)(z_2, y_2a_1 + x_2c_1, y_2b_1 + x_2d_1), M_1M_2) \\ &= (z_1 + z_2 + x_1y_2a_1 + x_1x_2c_1, y_1 + y_2a_1 + x_2c_1, x_1 + y_2b_1 + x_2d_1, M_1M_2)\end{aligned}\tag{41}$$

where $X_1 = (z_1, y_1, x_1) \in N, X_2 = (z_2, y_2, x_2) \in N, M_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in SL(2, \mathbb{R})$, and $M_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in SL(2, \mathbb{R})$

From now on, our useful for the multiplication in J will be as

$$(X_1, M_1)(X_2, M_2) = (X_1\rho(M_1)(X_2), M_1M_2)$$

Definition 4.1. Let $Q = H \times SL(2, \mathbb{R}) \rtimes_{\rho} SL(2, \mathbb{R})$ be the group with law:

$$\begin{aligned}X \cdot Y &= (X_1, M_1, M_2)(Y_1, N_1, N_2) \\ &= (X_1\rho(M_2)(Y_1), M_1 + N_1, M_2 + N_2)\end{aligned}\tag{43}$$

for all $X = (X_1, M_1, M_2) \in Q$ and $Y = (Y_1, N_1, N_2) \in Q$. From definition 4.1, the Jacobi group J can be identified with a subgroup $N \times \{I_{SL(2, \mathbb{R})}\} \rtimes_{\rho} SL(2, \mathbb{R})$ of Q . Let $A = N \times SL(2, \mathbb{R}) \times_{\rho} \{I_{SL(2, \mathbb{R})}\}$ be the subgroup of Q , which is the direct product of N with $SL(2, \mathbb{R})$

Definition 4.2. For any function $f \in \mathcal{D}(J)$, we can define a function \tilde{f} on Q by

$$\tilde{f}(X, M_1, M_2) = f(MX, M_1M_2)\tag{44}$$

Remark 4.1. The function \tilde{f} is invariant in the following sense

$$\tilde{f}(N^{-1}X, M_1, N^{-1}M_2) = \tilde{f}(N^{-1}X, M_1, N^{-1}M_2)\tag{45}$$

Theorem 4.1. For any function $\psi \in \mathcal{D}(J)$ and $\tilde{f} \in \mathcal{D}(Q)$ invariant in sense (32), we get

$$\psi * \tilde{f}(X, M_1, M_2) = \tilde{f} *_c \psi(X, M_1, M_2)\tag{46}$$

where $*$ signifies the convolution product on J with respect the variable (X, M_2) , and $*_c$ signifies the convolution product on A with respect the variable (X, M_1)

Proof : In fact we have

$$\begin{aligned}
& \psi * \tilde{f}(X, M_1, M_2) \\
&= \int_N \int_{SL(2, \mathbb{R})} \tilde{f}((Y, M)^{-1}(X, M_1, M_2)) \psi(Y, M) dY dM \\
&= \int_N \int_{SL(2, \mathbb{R})} \tilde{f}[(M^{-1}(-Y), M^{-1})(X, M_1, M_2)] \psi(Y, M) dY dM \\
&= \int_N \int_{SL(2, \mathbb{R})} \tilde{f}[(M^{-1}(Y - X), M_1, M^{-1}M_2)(X, M_1, M_2)] \psi(Y, M) dY dM \\
&= \int_N \int_{SL(2, \mathbb{R})} \tilde{f}[(M^{-1}(Y - X), M_1, M^{-1}M_2)] \psi(v', g') dY dM \\
&= \int_N \int_{SL(2, \mathbb{R})} \tilde{f}[Y - X, M_1 M^{-1}, M_2] \psi(Y, M) dY dM = \tilde{f} *_c \psi(X, M_1, M_2)
\end{aligned}$$

for any function $\psi \in \mathcal{D}(J)$ and $\tilde{f} \in \mathcal{D}(Q)$

Definition 4.3. For any $k_1 \in S^1$ let $\Gamma_{k_1} \Psi$ be the fuction defined by

$$\Gamma_{k_1} \Psi(v, g) = \Psi(X, gk_1) \quad (48)$$

for any $v \in N$, $g \in SL(2, \mathbb{R})$ and $k_1 \in S^1$

Definition 4.4. Let $f \in C_0^\infty(J)$, we define its Fourier transform by

$$\mathcal{F}_N T \mathcal{F} \Psi(\eta, m, \xi, \lambda) = \int_N \int_A \int_N \int_{S^1} \Psi(v, kna) e^{-i\langle \eta, v \rangle} e^{-ikm} a^{-i\lambda} e^{-i\langle \xi, n \rangle} dk da dn dv$$

where \mathcal{F}_N is the Fourier transform on N , $kna = g$, $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$, $v = (v_1, v_2, v_3) \in N$, and $dv = dv_1 dv_2 dv_3$ is the Lebesgue measure on N

$$\begin{aligned}
\langle \eta, v \rangle &= \langle (\eta_1, \eta_2, \eta_3), (v_1, v_2, v_3) \rangle \\
&= \eta_1 v_1 + \eta_2 v_2 + \eta_3 v_3
\end{aligned} \quad (49)$$

Plancherel's Theorem 4.2. For any function $f \in L^1(J) \cap L^2(J)$, we get

$$\int_J |\Psi(v, g)|^2 dv dg = \int_N \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} |\mathcal{F}_H T \mathcal{F} \Psi(\eta, m, \xi, \lambda)| d\eta d\lambda d\xi \quad (50)$$

Proof: For any function $\Psi \in L^1(J) \cap L^2(J)$, we get

$$\begin{aligned}
& \Gamma_{I_K} \Psi * \check{\Psi}(0, I_{SL(2, \mathbb{R})}, I_{SL(2, \mathbb{R})}) \\
&= \sum_{m=-\infty}^{\infty} [\Gamma_{k_1} \Psi * \check{\Psi}(0, I_{SL(2, \mathbb{R})}, I_{SL(2, \mathbb{R})}) e^{-ikm} dk_1] \\
&= \int_N \int_{SL(2, \mathbb{R})} \sum_{m=-\infty}^{\infty} \left[\int_{S^1} \check{\Psi}((w, g)^{-1}(0, I_{SL(2, \mathbb{R})}, I_{SL(2, \mathbb{R})})) \Gamma_{k_1} \Psi(w, g) e^{-ikm} dk_1 \right] dw dg \\
&= \int_N \int_{SL(2, \mathbb{R})} \sum_{m=-\infty}^{\infty} \left[\int_{S^1} \check{\Psi}(g^{-1}(0 - w), I_G, g^{-1} I_G) \Psi(w, k_1 g) e^{-ikm} dk_1 \right] dw dg \\
&= \int_N \int_{SL(2, \mathbb{R})} \sum_{m=-\infty}^{\infty} \left[\int_{S^1} \check{\Psi}(-w, I_G g^{-1}, I_G) \Psi(w, k_1 g) e^{-ikm} dk_1 \right] dw dg \\
&= \int_N \int_{SL(2, \mathbb{R})} \sum_{m=-\infty}^{\infty} \left[\int_{S^1} \check{\Psi}(g^{-1}(-w), g^{-1}) \Psi(w, k_1 g) e^{-ikm} dk_1 \right] dw dg \\
&= \int_N \int_{SL(2, \mathbb{R})} \check{\Psi}(g^{-1}(-w), g^{-1}) \Psi(w, I_{S^1} g) dw dg \\
&= \int_H \int_{SL(2, \mathbb{R})} \overline{\Psi[(g^{-1}(-w), g^{-1})^{-1}]} \Psi(w, g) dw dg \\
&= \int_H \int_{SL(2, \mathbb{R})} \overline{\Psi(w, g)} \Psi(w, g) dw dg \\
&= \int_H \int_{SL(2, \mathbb{R})} |\Psi(w, g)|^2 dw dg = \int_J |\Psi(w, g)|^2 dw dg
\end{aligned}$$

In other hand

$$\begin{aligned}
& \Gamma_{I_K} \Psi * \widetilde{\Psi}(0, I_G, I_G) \\
= & \int_H \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}(\Gamma_{k_1} \Psi * \widetilde{\Psi})(\eta, \xi, \lambda, m, I_G) e^{-ik_1 m} dk_1] d\eta d\xi d\lambda \\
= & \int_H \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}[\Gamma_{k_1} \Psi * \widetilde{\Psi}(v, I_k n a, I_G) \gamma(k_1^{-1}) dk_1] \\
& e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} dv dndad\eta d\xi d\lambda \\
= & \int_H \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}[\int_K \widetilde{\Psi}((w, g_2)^{-1}(v, I_k n a, I_G) \Gamma_{k_1} \Psi(w, g_2) \\
& \gamma(k_1^{-1})) dk_1] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} dv dndadwdg_2 d\eta d\xi d\lambda \\
= & \int_H \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}[\int_K \widetilde{\Psi}((g_2^{-1}(-w), g_2^{-1})(v, I_k n a, I_G) \Gamma_{k_1} \Psi(w, g_2) \\
& \gamma(k_1^{-1})) dk_1] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} dv dndadwdg_2 d\eta d\xi d\lambda \\
= & \int_G \int_{\mathbb{R}^4} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr}[\int_K \widetilde{\Psi}((g_2^{-1}(-w) + (g_2^{-1})(v), I_k n a, g_2^{-1} I_G) \\
& \Gamma_{k_1} \Psi(w, g_2) \gamma(k_1^{-1})) dk_1] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} dv dndadwdg_2 d\eta d\xi d\lambda \\
= & \int_N \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}[\int_K \widetilde{\Psi}((g_2^{-1}(v - w), I_k n a, g_2^{-1} I_G) \\
& \Psi(w, k_1 g_2) \gamma(k_1^{-1})) dk_1] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} dv dndadwdg_2 d\eta d\xi d\lambda \\
= & \int_N \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}[\int_K \widetilde{\Psi}(((v - w), n a g_2^{-1}, I_G) \Psi(w, k_1 g_2) \\
& \gamma(k_1^{-1})) dk_1] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} dv dndadwdg_2 d\eta d\xi d\lambda \\
= & \int_N \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}[\int_K \int_K \widetilde{\Psi}((v - w, I_k n a a_2^{-1} n_2^{-1} k_2^{-1}, I_G) \Psi(w, k_1 k_2 n_2 a_2) \\
& \gamma(k_1^{-1})) dk_1 dk_2] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} dv dndadwdn_2 da_2 d\eta d\xi d\lambda
\end{aligned}$$

We continue our calculation.

$$\begin{aligned}
& \Gamma_{I_K} \Psi * \widetilde{\Psi}(0, I_G, I_G) \\
&= \int_N \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}] \left[\int_K \int_K \widetilde{\Psi}((v, ank_2^{-1}, I_G) \Psi(w, k_1 k_2 n_2 a_2) \right. \\
&\quad \left. \gamma(k_1^{-1})) dk_1 dk_2 \right] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} e^{-i\langle w, \eta \rangle} e^{-i\langle n_2, \xi \rangle} a_2^{-i\lambda} dv dn da dw dn_2 da_2 d\eta d\xi d\lambda \\
&= \int_N \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}] \left[\int_K \int_K \widetilde{\Psi}((v, ank_2^{-1} k_1, I_G) \Psi(w, k_2 n_2 a_2) \gamma(k_1^{-1}) \gamma(k_2^{-1})) \right. \\
&\quad \left. dk_1 dk_2 \right] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} e^{-i\langle w, \eta \rangle} e^{-i\langle n_2, \xi \rangle} a_2^{-i\lambda} dv dn da dw dn_2 da_2 d\eta d\xi d\lambda \\
&= \int_N \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}] \left[\int_K \int_K \widetilde{\Psi}((v, ank_1, I_G) \Psi(w, k_2 n_2 a_2) \gamma(k_1^{-1}) \gamma(k_2^{-1})) \right. \\
&\quad \left. dk_1 dk_2 \right] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} e^{-i\langle w, \eta \rangle} e^{-i\langle n_2, \xi \rangle} a_2^{-i\lambda} dv dn da dw dn_2 da_2 d\eta d\xi d\lambda \\
&= \int_N \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}] \left[\int_K \int_K \widetilde{\Psi}(ank_1 v, ank_1) \Psi(w, k_2 n_2 a_2) \gamma(k_1^{-1}) \gamma(k_2^{-1}) \right. \\
&\quad \left. dk_1 dk_2 \right] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} e^{-i\langle w, \eta \rangle} e^{-i\langle n_2, \xi \rangle} a_2^{-i\lambda} dv dn da dw dn_2 da_2 d\eta d\xi d\lambda \\
&= \int_H \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}] \left[\int_K \int_K \overline{\Psi((ank_1(v), ank_1)^{-1})} \Psi(w, k_2 n_2 a_2) \gamma(k_1^{-1}) \right. \\
&\quad \left. \gamma(k_2^{-1})) dk_1 \right] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} e^{-i\langle w, \eta \rangle} e^{-i\langle n_2, \xi \rangle} a_2^{-i\lambda} dv dn da dw dk_2 dn_2 da_2 d\eta d\xi d\lambda \\
&= \int_H \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}] \left[\int_K \int_K \overline{\Psi(-v, k_1^{-1} n^{-1} a^{-1})} \Psi(w, k_2 n_2 a_2) \gamma(k_1^{-1}) \gamma(k_2^{-1}) \right. \\
&\quad \left. dk_1 dk_2 \right] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} e^{-i\langle w, \eta \rangle} e^{-i\langle n_2, \xi \rangle} a_2^{-i\lambda} dv dn da dw dn_2 da_2 d\eta d\xi d\lambda \\
&= \int_H \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} [\mathcal{F}_H T \mathcal{F}] \left[\int_K \int_K \overline{\Psi(v, k_1 n a)} \Psi(w, k_2 n_2 a_2) \gamma^*(k_1^{-1}) \gamma(k_2^{-1}) \right. \\
&\quad \left. dk_1 dk_2 \right] e^{-i\langle v, \eta \rangle} e^{-i\langle n, \xi \rangle} a^{-i\lambda} e^{-i\langle w, \eta \rangle} e^{-i\langle n_2, \xi \rangle} a_2^{-i\lambda} dv dn da dw dn_2 da_2 d\eta d\xi d\lambda \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} [\mathcal{F}_{\mathbb{R}^4} T \mathcal{F} \overline{\Psi}(\eta, \gamma^*, \xi, \lambda) \mathcal{F}_{\mathbb{R}^4} T \mathcal{F} \Psi(\eta, \gamma, \xi, \lambda)] d\eta d\xi d\lambda \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \sum_{\gamma \in \widehat{K}} d_\gamma \|\mathcal{F}_{\mathbb{R}^4} T \mathcal{F} \Psi(\eta, \gamma, \xi, \lambda)\|_{H.S}^2 d\eta d\xi d\lambda
\end{aligned}$$

5 Left Ideals of the Group Algebra $L^1(N)$.

First, I will prove the solvability of any invariant differential operator on the connected solvable group $N = \mathbb{R}^2 \rtimes_{\sigma} \mathbb{R}$. Therefor, I will extend the group by a

larger group $E = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$, with multiplication (n, a, b) and (m, x, y) as

$$(n, a, b)(m, x, y) = (n + m + \sigma(b)x, a + x, b + y) \quad (51)$$

Let $\mathbb{F} = \mathbb{R}^2 \times \mathbb{R}$ be the abelian group, which is the direct product of two real vector groups \mathbb{R}^2 and \mathbb{R}

Definition 5.1. For any function $f \in \mathcal{D}(N)$, we can define a function τf on E by

$$\tau f(n, a, b) = f(\sigma(a)n, ab) \quad (52)$$

Remark 5.1. The function τf is invariant in the following sense

$$\tau f(\sigma(x^{-1})n, xa, x^{-1}b) = \tau f(n, a, b) \quad (53)$$

Therefor denote by $\tau C^\infty(N)$ (resp. $\tau C^\infty(\mathbb{F})$) the image of $C^\infty(N)$ (resp. $C^\infty(\mathbb{F})$) then we have

$$\begin{aligned} \tau C^\infty(N)|_N &= C^\infty(N) \\ \tau C^\infty(\mathbb{F})|_{\mathbb{F}} &= C^\infty(\mathbb{F}) \end{aligned} \quad (54)$$

Definition 5.2. Let be the mapping $\Lambda : \tau C^\infty(E)|_{\mathbb{F}} \longrightarrow \tau C^\infty(E)|_N$ defined by

$$\Lambda(\tau f|_{\mathbb{F}})(z, y, 0) = \tau f|_N(z, 0, y) \quad (55)$$

is topological isomorphisms and its inverse is nothing but Γ^{-1} defined by

$$\Lambda^{-1}(\tau f|_N)(n, 0, a) = \tau f|_{\mathbb{F}}(n, a, 0) \quad (56)$$

My main result is

Theorem 5.1. If P_u any invariant differential operator on N associated to the distribution $u \in \mathcal{U}$, then, we have

$$P_u C^\infty(N) = C^\infty(N) \quad (57)$$

Proof: Let Q_u be the invariant differential operator with constant coefficients on K associated to u , then by the theory of differential operators with constant coefficients [20], we get

$$Q_u \tau C^\infty(E)|_{\mathbb{F}} = \tau C^\infty(E)|_{\mathbb{F}} = C^\infty(\mathbb{F}) \quad (58)$$

That means for any $\psi(n, a) \in C^\infty(\mathbb{F})$, there exist a function $\varphi(n, a, x) \in \tau C^\infty(E)|_{\mathbb{F}}$, such that

$$Q_u \varphi(n, a, 1) = u *_c \varphi(n, a, 0) = \psi(n, a) \quad (59)$$

The function $\psi(n, a)$ can be transformed as an invariant function $\psi \in \tau C^\infty(E)|_{\mathbb{F}}$ as follows

$$\psi(n, a) = \tau \psi(\rho(a^{-1})n, a, 0) \quad (60)$$

In other side, we have

$$\begin{aligned}
& \Lambda Q_u \varphi(n, a, 0) \\
&= Q_u \varphi(n, 0, a) = u *_c \varphi(n, 0, a) \\
&= u * \varphi(n, 1, a) = P_u \varphi(n, 0, a) \\
&= \Lambda \tau \psi(\rho(a^{-1})n, a, 1) = \tau \psi(\sigma(a^{-1})n, 1, a) \\
&= \psi(n, a)
\end{aligned} \tag{61}$$

So the proof of the solvability of any right invariant differential operator on N .

If I is a subspace of $L^1(N)$, we denote τI its image by the mapping τ , let $J = \tau I|_{\mathbb{F}}$. My main result is:

Theorem 5.2. *Let I be a subspace of $L^1(N)$, then the following conditions are equivalents.*

- (i) $J = \tau I|_{\mathbb{F}}$ is an ideal in the Banach algebra $L^1(\mathbb{F})$.
- (ii) I is a left ideal in the Banach algebra $L^1(N)$.

Proof: (i) implies (ii) Let I be a subspace of the space $L^1(N)$ and τI the image of I by τ such that $J = \tau I|_{\mathbb{F}}$ is an ideal in $L^1(\mathbb{F})$, then we have:

$$u *_c \tau I|_{\mathbb{F}}(n, a, 0) \subseteq \tau I|_{\mathbb{F}}(n, a, 0) \tag{62}$$

for any $u \in L^1(\mathbb{F})$ and $(n, a) \in \mathbb{F}$, where

$$u *_c \tau I|_{\mathbb{F}}(n, a, 0) = \left\{ \int_{\mathbb{F}} \tau f|_{\mathbb{F}} [n - m, a - b, 0] u(m, b) dm \frac{db}{b}, f \in I \right\} \tag{63}$$

It shows that

$$u *_c \tau f|_{\mathbb{F}}(n, a, 0) \in \tau I|_{\mathbb{F}}(n, a, 0) \tag{64}$$

for any $\tau f \in \tau I$. Apply equation(32), we get

$$\begin{aligned}
& \Gamma(u *_c \tau f|_{\mathbb{F}})(n, a, 0) \\
&= u * \tau f(n, 1, a) \in \Gamma(\tau I|_{\mathbb{F}}(n, a, 0)) \\
&= \tau I|_N(n, 0, a) = I
\end{aligned} \tag{65}$$

(ii) implies (i), if I is an ideal in $L^1(N)$, then we get

$$\begin{aligned}
& u * \tau I|_N(n, 0, a) \\
&= u * I(n, a) \subseteq \tau I|_N(n, 0, a) = I(n, a)
\end{aligned} \tag{66}$$

where

$$u * \tau I|_N(n, 1, a) = \left\{ \int_N \tau f|_N [\sigma(-b)(n - m), 1, a - b] u(m, b) dm \frac{db}{b}, f \in I \right\} \tag{67}$$

By equation (36), we obtain

$$\begin{aligned}
& \chi^{-1}(u * \tilde{f} |_N)(n, 0, a) \\
&= u *_c \tilde{f} |_F(n, a, 0) \in \chi^{-1}(u * \tilde{I} |_N)(n, a, 0) \\
&= u * \tilde{I} |_F(n, a, 0)
\end{aligned} \tag{68}$$

Corollary 5.1. *Let I be a subspace of the space $L^1(N)$ and τI its image by the mapping τ such that $J = \tau I|_F$ is an ideal in $L^1(F)$, then the following conditions are verified.*

(1) *J is a closed ideal in the algebra $L^1(F)$ if and only if I is a left closed ideal in the algebra $L^1(N)$.*

(2) *J is a prime ideal in the algebra $L^1(F)$ if and only if I is a left prime ideal in the algebra $L^1(N)$.*

(3) *J is a maximal ideal in the algebra $L^1(F)$ if and only if I is a left maximal ideal in the algebra $L^1(N)$.*

(4) *J is a dense ideal in the algebra $L^1(F)$ if and only if I is a left dense ideal in the algebra $L^1(N)$.*

The proof of this corollary results immediately from theorem 5.2.

6 Left Ideals of the Group Algebra $L^1(N \rtimes S)$.

Let $S = SL(2, \mathbb{R})/SO(2)$ the symmetric space of the real semi simple Lie group $SL(2, \mathbb{R})$, which is diffeomorphism on the group

$$S = SL(2, \mathbb{R})/SO(2) = \left\{ X = \begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{R}_+^* \right\} \tag{69}$$

The group S is isomorphic onto the group $\mathbb{R} \rtimes_{\varrho} \mathbb{R}_+^*$ semidirect product of the two group \mathbb{R} and \mathbb{R}_+^* where $\varrho : \mathbb{R}_+^* \rightarrow Aut(\mathbb{R})$ is the group homomorphism from the real group into the group $Aut(\mathbb{R})$ of all automorphisms of the vector group \mathbb{R} , defined as

$$\varrho(x)(n) = xn$$

First, I will prove the solvability of any invariant differential operator on the connected solvable group S . Therefor denote by $W = \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}_+^*$, with the following law defined as

$$(n, x, y)(m, a, b) = (n + \varrho(y)m, xa, yb) = (n + m + y^2a, xa, yb) \tag{70}$$

for any $(n, a) \in S$, and $(m, b) \in S$, here $\varrho(a)m = a^2m$. Let K be the group $\mathbb{R} \times \mathbb{R}_+^*$, which is the direct product of the group \mathbb{R} with the group \mathbb{R}_+^* . So the group S can be identified with the subgroup $\mathbb{R} \times \{1\} \times \mathbb{R}_+^*$ of W and K can be identified with the subgroup $\mathbb{R} \times \mathbb{R}_+^* \times \{1\}$ of W .

Definition 6.1. For any function $f \in \mathcal{D}(S)$, we can define a function τf on W by

$$\tau f(n, a, b) = f(\varrho(a)n, ab) \quad (71)$$

Remark 6.1. The function τf is invariant in the following sense

$$\tau f(\varrho(x^{-1})n, xa, x^{-1}b) = \tau f(n, a, b) \quad (72)$$

Now denote by $\tau(C^\infty(S))$ (resp. $\tau(C^\infty(K))$) the image of $C^\infty(S)$ (resp. $C^\infty(K)$) by the transformation τ , then we have

$$\begin{aligned} \tau(C^\infty(S))|_S &= C^\infty(S) \\ \tau(C^\infty(K))|_K &= C^\infty(K) \end{aligned} \quad (73)$$

Definition 6.2. Let be the mapping $\chi : \tau(C^\infty(K))|_K \longrightarrow \tau(C^\infty(S))|_S$ defined by

$$\tau f|_K(z, y, 1) \rightarrow \tau f|_N(z, 1, y) \quad (74)$$

$$\tau f|_K(n, a, 1) \rightarrow \tau f|_S(n, 1, a) \quad (75)$$

is topological isomorphisms and its inverse is nothing but χ^{-1} defined by

$$\tau f|_S(n, 1, a) \rightarrow \tau f|_K(n, a, 1) \quad (76)$$

My main result is

Theorem 6.1. If P_u any invariant differential operator on S associated to the distribution $u \in \mathcal{U}$, then, we have

$$P_u C^\infty(S) = C^\infty(S) \quad (77)$$

Proof: Let Q_u be the invariant differential operator with constant coefficients on K associated to u , then by the theory of differential operators with constant coefficients [20], we get

$$Q_u \tau(C^\infty(K))|_K = \tau(C^\infty(K))|_K = C^\infty(K) \quad (78)$$

That means for any $\psi(n, a) \in C^\infty(K)$, there exist a function $\varphi(n, a, x) \in \tau(C^\infty(K))|_K$, such that

$$Q_u \varphi(n, a, 1) = u *_c \varphi(n, a, 1) = \psi(n, a) \quad (79)$$

The function $\psi(n, a)$ can be transformed as an invariant function $\psi \in \tau(C^\infty(K))|_K$ as follows

$$\psi(n, a) = \tau \psi(\varrho(a^{-1})n, a, 1) \quad (80)$$

In other side, we have

$$\begin{aligned}
& \chi Q_u \varphi(n, a, 1) \\
&= Q_u \varphi(n, 1, a) = u *_c \varphi(n, 1, a) \\
&= u * \varphi(n, 1, a) = P_u \varphi(n, 1, a) \\
&= \chi \tau \psi(\varrho(a^{-1})n, a, 1) = \tau \psi(\varrho(a^{-1})n, 1, a) \\
&= \psi(n, a)
\end{aligned} \tag{81}$$

So the proof of the solvability of any right invariant differential operator on S .

If I is a subspace of $L^1(S)$, we denote by τI its image by the mapping τ , let $\omega = \tau I|_K$. My main result is:

Theorem 6.2. *Let I be a subspace of $L^1(S)$, then the following conditions are equivalents.*

- (i) $\omega = \tau I|_K$ is an ideal in the Banach algebra $L^1(K)$.
- (ii) I is a left ideal in the Banach algebra $L^1(S)$.

Proof: (i) implies (ii) Let I be a subspace of the space $L^1(S)$ such that $\omega = \tau I|_K$ is an ideal in $L^1(K)$, then we have:

$$u *_c \tau I|_K(n, a, 1) \subseteq \tau I|_K(n, a, 1) \tag{82}$$

for any $u \in L^1(K)$ and $(n, a) \in K$, where

$$u *_c \tau I|_K(n, a, 1) = \left\{ \int_K \tau f|_K [n - m, a - b, 1] u(m, b) dm \frac{db}{b}, f \in I \right\} \tag{83}$$

It shows that

$$u *_c \tau f|_K(n, a, 1) \in \tau I|_K(n, a, 1) \tag{84}$$

for any $\tau f \in \tau I$. According to equation(82), we get

$$\begin{aligned}
& \chi(u *_c \tau f|_K)(n, a, 1) \\
&= u * \tau f(n, 1, a) \in \chi(\tau I|_K)(n, a, 1) \\
&= \tau I|_S(n, 1, a) = I(n, a)
\end{aligned} \tag{85}$$

(ii) implies (i), if I is an ideal in $L^1(S)$, then we get

$$\begin{aligned}
& u * \tau I|_S(n, 1, a) \\
&= u * I(n, a) \subseteq \tau I|_S(n, 1, a) = I(n, a)
\end{aligned} \tag{86}$$

where

$$u * \tau I|_S(n, 1, a) = \left\{ \int_S \tau f|_S [\rho(-b)(n - m), 1, a - b] u(m, b) dm \frac{db}{b}, f \in I \right\} \tag{87}$$

By equation (36), we obtain

$$\begin{aligned}
& \chi^{-1}(u * \tau f|_S)(n, 1, a) \\
&= u *_c \tau f|_K(n, a, 1) \in \chi^{-1}(u * \tau I|_S)(n, a, 1) \\
&= u * \tau I|_K(n, a, 1)
\end{aligned} \tag{88}$$

Corollary 6.1. *Let I be a subspace of the space $L^1(S)$ and τI its image by the mapping τ such that $\omega = \tau I|_K$ is an ideal in $L^1(K)$, then the following conditions are verified.*

(1) ω is a closed ideal in the algebra $L^1(K)$ if and only if I is a left closed ideal in the algebra $L^1(S)$.

(2) ω is a prime ideal in the algebra $L^1(K)$ if and only if I is a left prime ideal in the algebra $L^1(S)$.

(3) ω is a maximal ideal in the algebra $L^1(K)$ if and only if I is a left maximal ideal in the algebra $L^1(S)$.

(4) ω is a dense ideal in the algebra $L^1(K)$ if and only if I is a left dense ideal in the algebra $L^1(S)$.

The proof of this corollary results immediately from theorem 6.2.

The Heisenberg group N is the semi-direct product of the two vector Lie group $\mathbb{R}^2 \rtimes_{\sigma} \mathbb{R}$. I extend the group $M = N \times S$ by considering the new group $V = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times S$ with the following law

$$\begin{aligned}
& X \cdot Y \\
&= (n_3, n_2, n_1, n_4, a_1, a_2, a_3)(m_3, m_2, m_1, m_4, b_1, b_2, b_3) \\
&= ((n_3 + m_3 + \sigma(n_4)(m_3, m_2), n_2 + m_2, n_1 + m_1, n_4 + m_4), (a_1 b_1, a_2 b_2, a_3 b_3)) \\
&= ((n_3 + m_3 + n_4 m_2, n_2 + m_2, n_1 + m_1, n_4 + m_4), (a_1 b_1, a_2 b_2, a_3 b_3))
\end{aligned} \tag{89}$$

Denote by $B = \mathbb{R}^2 \times \mathbb{R} \times S$ the commutative Lie group of the direct product of three Lie groups \mathbb{R}^2 , \mathbb{R} , and S . In this case the group $M = N \times S$ can be identified with the sub-group $\mathbb{R}^2 \times \{0\} \times \mathbb{R} \times S$ and the group $B = \mathbb{R}^2 \times \mathbb{R} \times S$ can be identified with the sub-group $\mathbb{R}^2 \times \mathbb{R} \times \{0\} \times S$.

Definition 6.3. *Any function $\psi \in C^\infty(M)$ can be extended to a unique function $\Xi\psi$ belongs to $C^\infty(V)$, as follows*

$$\begin{aligned}
& \Xi\psi((n_3, n_2, n_1, n_4), s) \\
&= \psi((\sigma(n_1)(n_3, n_2), n_1 + n_4), s) \\
&= \psi((n_1(n_3, n_2), n_1 + n_4), s) \\
&= \psi((n_3 + n_1 n_2, n_2, n_1 + n_4), s)
\end{aligned} \tag{90}$$

for any $(n_3, n_2, n_1, n_4) \in N \times \mathbb{R}$, $s \in S$, $n_1(n_3, n_2) = (n_3 + n_1 n_2, n_2) = \sigma(n_1)(n_3, n_2)$

If I is a subspace of $L^1(M)$, we denote ΞI its image by the mapping Ξ . Let $J = \Xi I|_B$.

My main result is:

Theorem 6.3. *Let I be a subspace of $L^1(K)$, then the following conditions are equivalents.*

(i) $J = \tilde{I}|_B$ is an ideal in the Banach algebra $L^1(B)$.

(ii) I is a left ideal in the Banach algebra $L^1(M)$.

For the proof of this theorem, I refer to my book [9, Chap I, theorem 3.1.]

Corollary 6.2. *Let I be a subspace of the space $L^1(M)$ and ΞI its image by the mapping Ξ such that $J = \Xi I|_B$ is an ideal in $L^1(B)$, then the following conditions are verified.*

(1) J is an ideal in the algebra $L^1(B)$ if and only if I is a closed ideal in the algebra $L^1(M)$ if and only if I is a closed left ideal in the algebra $L^1(N \rtimes S)$.

(2) J is a prime ideal in the algebra $L^1(B)$ if and only if I is a prime ideal in the algebra $L^1(M)$ if and only if I is a prime left ideal in the algebra $L^1(N \rtimes S)$

(3) J is a maximal ideal in the algebra $L^1(B)$ if and only if I is a maximal ideal in the algebra $L^1(M)$ if and only if I is a left maximal ideal in the algebra $L^1(N \rtimes S)$

(4) J is a dense ideal in the algebra $L^1(B)$ if and only if I is a dense ideal in the algebra $L^1(M)$ if and only if I is a left dense ideal in the algebra $L^1(N \rtimes S)$

For the proof of this theorem, I refer to **Theorem 6.2.** and **Corollary 6.1.**

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